

Quantum and classical fidelities for Gaussian states

Hyunseok Jeong and Timothy C. Ralph

Department of Physics, University of Queensland, St. Lucia, Queensland 4072, Australia

Warwick P. Bowen

The Jack Dodd Centre for Photonics and Ultra Cold Atoms, Department of Physics, University of Otago, Dunedin, New Zealand

Received May 2, 2006; revised July 14, 2006; accepted July 18, 2006;
posted July 24, 2006 (Doc. ID 70459); published January 26, 2007

We examine the physical significance of fidelity as a measure of similarity for Gaussian states by drawing a comparison with its classical counterpart. We find that the relationship between these classical and quantum fidelities is not straightforward, and in general does not seem to provide insight into the physical significance of quantum fidelity. To avoid this ambiguity we propose that the efficacy of quantum information protocols be characterized by determining their transfer function and then calculating the fidelity achievable for a hypothetical pure reference input state. © 2007 Optical Society of America

OCIS codes: 270.0270, 350.4800.

1. INTRODUCTION

Quantification of the similarity (or distinguishability) of quantum states is a crucial issue in quantum information theory.¹ Quantum fidelity²—previously known as Uhlmann’s transition probability³—is probably the most well known such quantification technique, and is an important tool for assessing the efficacy of quantum information transfer.⁴ Critical to any technique used to characterize similarity is a robust understanding of its physical significance. To date, although there have been efforts to impose an operational interpretation on quantum fidelity for mixed states⁵ and to compare it with alternative distance measures,^{6,7} a strong and general physical significance has yet to be found. When one of the states involved is pure, it is well known that quantum fidelity is equal to the transition probability from one state to the other. Furthermore, Uhlmann’s theorem allows the quantum fidelity between two arbitrary states to be translated to a fidelity between higher-dimensional pure states, which can then be interpreted as a transition probability between those higher-dimensional states.³ However, the strength of the link between these hypothetical higher-dimensional states and the actual states under investigation is not obvious. In this paper we seek to establish the physical significance of quantum fidelity for a particular class of states, those of a Gaussian nature.

Gaussian states are extremely useful tools in many quantum optics experiments. For example, several quantum communication experiments have now been performed using only Gaussian states.^{8–12} Typically the formulas used to calculate the quantum fidelity achieved by these experiments assume that the input states are pure coherent states.^{8–12} There have been many studies of quantum fidelity as a successful criteria for quantum teleportation of coherent states,^{4,13–15} and its value in this re-

gime is well understood. However, the unknown quantum states supplied by a third party called “Victor” in real experiments are not perfectly pure, and typically have some small but nonnegligible level of mixedness. It is important to understand both the effect of this mixedness on the quantum fidelity achieved by experiments and the significance of the resulting fidelity results, which turn out to be markedly different from those expected for a coherent state even for extremely small levels of mixedness. Hence the motivation for this paper.

Quantum fidelity is a direct extension of the fidelity between a pair of classical probability distributions, termed here classical fidelity, which is used in statistics to characterize their similarity. For Gaussian states, in particular, this relationship is interesting, since the Wigner function describing such states is nonnegative and can be thought of, to some degree, as a classical probability distribution, which we shall discuss more rigorously in this paper. It might then be expected that the quantum and classical fidelities would coincide for Gaussian states, and thus a robust physical significance could be established for quantum fidelity. The results reported here show, however, that this is not the case.

In this paper, we compare and contrast quantum and classical fidelities as quantum–classical counterparts. The explicit forms of the quantum fidelities between two Gaussian states are obtained for various cases. We also point out that the classical fidelity between two Gaussian states inferred by the complementary measurements exactly corresponds to the overlap between their Wigner functions. Then, although the quantum and classical fidelities do coincide in the classical limit, i.e., in the limit of the extreme mixedness, we show that no simple relationship could be established for nonmaximally mixed Gaussian states. The mixedness, squeezing, and separa-

tion of the Gaussian states involved each effect the discrepancy between classical and quantum fidelity in entirely different manners.

The unclear physical significance of quantum fidelity between mixed states raises questions about its usefulness as a measure of the efficacy of quantum information protocols. We propose a new characterization method to avoid this issue. In this method, the transfer function of the quantum information protocol is determined and used to characterize the quantum fidelity achievable for an arbitrary pure input state. This method has two advantages: (1) it provides a standard benchmark through which to compare different experiments; and (2) the input state can be chosen to ensure that one of the states used to determine the quantum fidelity is pure, yielding a physically significant fidelity, which represents the transition probability from one state to the other.

2. MOTIVATION

As we have noted in Section 1, the unknown quantum states used in real continuous-variable quantum teleportation experiments^{8–11} are not exactly pure states but have some small level of mixedness. In most experiments, the input states have been assumed pure in assessing the efficacy of the quantum information protocols without justification (we shall further clarify and discuss this point in Section 4). A natural question here is, how sensitive is fidelity to small levels of mixedness? It is generally assumed that the small level of mixedness involved will not significantly change the fidelity between the input and output states. However, as we will see here, in general this turns out not to be the case.

Let us consider the no-entanglement fidelity limit for unity gain quantum teleportation of coherent states. This is generally accepted to be given by a protocol in which Alice makes an ideal heterodyne measurement of the unknown state, thus obtaining an estimate of its coherent amplitude α , and then passes the measurement result to Bob who displaces a local vacuum mode by this estimated value.⁸ Bob's state then has the same average coherent amplitude as the unknown state, but its variance is increased by two units of vacuum noise. If the input state is a pure coherent state, then the no-entanglement fidelity is $F_{\text{ne}}=0.5$. Both the fidelity of quantum teleportation and the no-entanglement fidelity limit depend on the class of input states used. The degradation of fidelity in no-entanglement quantum teleportation is a result of noise introduced to the output state during the measurement and reconstruction processes. In the case when the input states are pure, this noise causes a significant change to the Wigner function describing the output state, with the result of poor overlap, fidelity, between the input and output states. However, as the input states become more and more thermal, and hence their breadths become larger, the noise plays a smaller role in the overlap between input and output states. In the limit that the magnitude of noise introduced is insignificant compared to the breadth of the input state, the overlap between the input and output states, and hence the no-entanglement fidelity limit, approaches unity. Our question in this section is how rapid this transition from 0.5 to 1 is as the input states become thermalized.

A general expression for the fidelity in this situation can easily be obtained by invoking Uhlmann's theorem.³ It states that if $\rho_1=\text{Tr}_a\{|\psi_1\rangle\langle\psi_1|\}$ and $\rho_2=\text{Tr}_a\{|\psi_2\rangle\langle\psi_2|\}$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are pure two-mode states and the partial traces are taken only over one of the modes, then the fidelity of ρ_1 with respect to ρ_2 is given by

$$F_{\rho_1,\rho_2}=\max_{|\psi_{1,2}\rangle}|\langle\psi_2\|\psi_1\rangle|^2, \quad (1)$$

where the maximization is over all pure states that have the required reduced density operators. In our case, the reduced density operators must be those describing the input and output states, both of which are isotropically mixed coherent states. Here we limit our analysis to unity gain teleportation, in which case the average coherent amplitudes of the input and output states are equal. We can then choose $\alpha_1=\alpha_2=0$ without loss of generality, and the input and output states then become thermal. In general, to calculate the fidelity between input and output states we must then maximize the fidelity over all higher-dimensional states that have reduced density operators describing the required thermal states. However, it is well known that the partial trace over a two-mode squeezed state, given by

$$|\phi_i\rangle=\frac{1}{G_i}\sum_n\left[\frac{(G_i-1)}{G_i}\right]^{n/2}|n\rangle_a|n\rangle_b, \quad (2)$$

where $G_i\geq 1$ is the strength of the squeezing, results in a thermal state.¹⁶ From symmetry it is clear that such a choice will maximize the fidelity as required. Thus the fidelity between two thermal states is given by

$$\begin{aligned} F &= |\langle\phi_1\|\phi_2\rangle|^2 = \frac{1}{G_1G_2}\left\{\sum_n\left[\frac{(G_1-1)(G_2-1)}{G_1G_2}\right]^{n/2}\right\}^2 \\ &= \left[\frac{1}{\sqrt{G_1G_2}-\sqrt{(G_1-1)(G_2-1)}}\right]^2 \\ &= \left[\frac{2}{\sqrt{(V_1+1)(V_2+1)}-\sqrt{(V_1-1)(V_2-1)}}\right]^2, \end{aligned} \quad (3)$$

where $V_i=2G_i-1$ is the variance of the single-mode thermal state obtained by the partial trace of the corresponding two-mode squeezed state.

We are now in a position to calculate the no-entanglement fidelity limit for teleportation of an isotropically mixed coherent state. Using Eq. (3) and the fact that the ideal heterodyne protocol discussed above adds two units of vacuum noise to the output, we obtain

$$F_{\text{ne}}=\left[\frac{2}{\sqrt{(V+1)(V+3)}-\sqrt{(V-1)(V+1)}}\right]^2. \quad (4)$$

This expression is graphed as a function of the input variance $V\geq 1$ in Fig. 1. Notice that we recover $F_{\text{ne}}=0.5$ at $V=1$, i.e., a pure coherent state, but that the fidelity is very sensitive to small amounts of mixedness. For example, for 2% mixedness, i.e., $V=1.02$, a typical level of experimental purity, we have $F_{\text{ne}}=0.57$, a 14% increase in the classical limit. We see, therefore, that to ensure accurate fidelity results, it is critical that the analysis of quantum teleportation experiments takes into account the mixed-

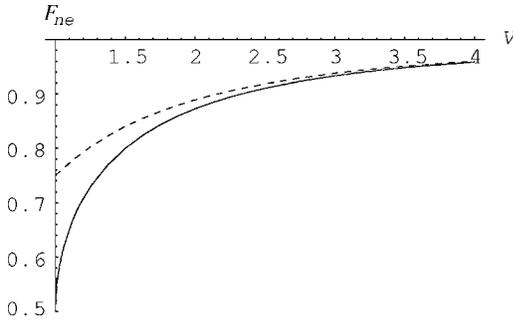


Fig. 1. Fidelity limit F_{ne} for quantum teleportation without entanglement against the variance V of an isotropically mixed-state input (solid curve) and the corresponding classical fidelity (dashed curve).

ness of the input state. Just as important as achieving an accurate fidelity result is to understand the actual significance of the result. One way of doing this is to consider the analogous classical system. The comparison of quantum and classical fidelities is the topic of Section 3, and the classical fidelity [see Eq. (11)] between probability distributions equivalent to the Wigner functions of the teleporter input and output states is also plotted in Fig. 1. Notice that for low levels of mixedness the classical and quantum fidelities are in stark contrast, but as the mixedness increases they asymptote to the same value.

3. COMPARISON BETWEEN QUANTUM AND CLASSICAL FIDELITIES FOR GAUSSIAN DISTRIBUTIONS

A. General Expressions for Fidelities of Gaussian States
Classical fidelity F_c and quantum fidelity F_q are defined as¹

$$F_c(P_1, P_2) = \left[\int d^2\alpha \sqrt{P_1(\alpha)P_2(\alpha)} \right]^2, \quad (5)$$

$$F_q(\rho_1, \rho_2) = \{\text{Tr}[\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}]\}^2, \quad (6)$$

where P_1 and P_2 are probability distributions, and ρ_1 and ρ_2 are density matrices. These density matrices can be equivalently represented as quasi-probability distributions, such as the Wigner function. The Wigner function of a general Gaussian state is

$$W(\alpha) = \frac{2}{\pi\sqrt{V^+V^-}} \exp \left[-\frac{2}{V^+}(\alpha_r \cos \phi + \alpha_i \sin \phi - \delta_r)^2 - \frac{2}{V^-}(\alpha_i \cos \phi - \alpha_r \sin \phi - \delta_i)^2 \right], \quad (7)$$

where $V^+ = [\Delta X(\phi)]^2$, $V^- = [\Delta P(\phi)]^2$, $\hat{X}(\phi) = (e^{-i\phi}\hat{a} + e^{i\phi}\hat{a}^\dagger)/2$, and $\hat{P}(\phi) = -i(e^{-i\phi}\hat{a} - e^{i\phi}\hat{a}^\dagger)/2$. Note that the variances V^\pm are directly measurable values in experiments. Equation (7) becomes a coherent state of amplitude $\delta = \delta_r + i\delta_i$ when $V^+ = V^- = 1$. Here the breadth of the distribution is quantified by the product V^+V^- . For quantum states this corresponds directly to the mixedness of the state, where $V^+V^- = 1$ for a pure state and $V^+V^- \rightarrow \infty$ as the mixedness increases. The level of squeezing of a Gaussian state is de-

termined by the squeezing parameter r :

$$r = \frac{1}{4} e^{i\phi} \ln \left[\frac{V^-}{V^+} \right]. \quad (8)$$

We will compare two Gaussian distributions labeled by the subscripts 1 and 2. These Gaussian distributions correspond to Wigner functions for quantum fidelity and to probability distributions for classical fidelity.

Suppose an ensemble of a Gaussian quantum state ρ . One can measure $\hat{X}(\tilde{\phi})$ many times while varying the angle $\tilde{\phi}$ to find the squeezed angle ϕ . Furthermore, the accumulated measurement results for $\hat{X}(\phi)$ and $\hat{P}(\phi)$ will result in the Gaussian probability distributions $P(\alpha'_r)$ and $P(\alpha'_i)$, where

$$\alpha'_r = \alpha_r \cos \phi + \alpha_i \sin \phi, \quad (9)$$

$$\alpha'_i = \alpha_i \cos \phi - \alpha_r \sin \phi, \quad (10)$$

with variances V^+ and V^- . A classical probability distribution $P(\alpha')$ for α_r and α_i constructed from $P(\alpha'_r)$ and $P(\alpha'_i)$ will be identical to the Wigner function $W(\alpha)$ in Eq. (7), which can also be represented by $W(\alpha')$ with the rotated variable α' . In other words, if one infers the two-dimensional classical probability distribution from the repetitive complementary measurements on the ensemble of the Gaussian quantum state ρ , it will be exactly the same as the Wigner function of state ρ . Therefore, the Wigner functions should be used to calculate the classical fidelity for Gaussian states. It should be noted that other quasi-probability distributions such as the P function or Q function cannot be used for this purpose. Of course, this approach cannot be generally applied to non-Gaussian states that may have negativity in the Wigner functions. A similar treatment can be found in a recent study,¹⁷ where the author showed that a good estimate of the fidelity of a quantum process can be obtained by measuring the outputs for only two complementary sets of input states. In short, the Wigner function of a Gaussian state is a good analogy of the classical probability distribution in the phase space, and Eq. (5) with the Wigner functions can be a reasonable measure of the classical similarity of two Gaussian states. On the other hand, it is nontrivial to represent quantum fidelity between mixed states in terms of their Wigner functions. However, when one of the states is pure, quantum fidelity in Eq. (6) can be expressed in terms of the Wigner functions as $F_q(\rho_1, \rho_2) = \pi \int d^2\alpha W_1(\alpha)W_2(\alpha)$.¹⁸⁻²⁰ It is interesting to note that this formula is obviously different from classical fidelity in Eq. (5).

Let us first consider the case where $\delta_1 = \delta_2$, i.e., the two Gaussian distributions have the same center. If we interpret the Wigner function [Eq. (7)] as a probability distribution, the classical fidelity between the Gaussian distributions $P_1(\alpha)$ and $P_2(\alpha)$ is straightforwardly obtained by Eq. (5) as

$$F_c = 4 \sqrt{V_1^+ V_1^- V_2^+ V_2^-} \{ \cos^2 \varphi (V_1^+ + V_2^+) (V_1^- + V_2^-) + \sin^2 \varphi (V_1^+ + V_2^-) (V_1^- + V_2^+) \}^{-1}, \quad (11)$$

where $\varphi = \phi_2 - \phi_1$ is the angle between the Gaussian dis-

tributions (see Fig. 2). The quantum fidelity between Gaussian states has been studied by several authors in previous work and some useful analytical expressions have been found.²¹⁻²⁴ It is possible to transform their formulas into more experimentally useful forms in terms of $V_{1,2}^{\pm}$ and φ . The quantum fidelity is found to be (see Appendix A)

$$F_q = \frac{2}{\sqrt{4\sqrt{V_1^+V_2^+V_1^-V_2^-}/F_c + K} - \sqrt{K}}, \quad (12)$$

where $K=(V_1^+V_1^- - 1)(V_2^+V_2^- - 1)$. For most of the cases considered in this paper, the angle φ is zero; in this case the classical and quantum fidelities are

$$F_c(\varphi = 0) = \frac{4\sqrt{V_1^+V_1^-V_2^+V_2^-}}{(V_1^+ + V_2^+)(V_1^- + V_2^-)}, \quad (13)$$

$$F_q(\varphi = 0) = 2\{\sqrt{(V_1^+V_2^- + 1)(V_1^-V_2^+ + 1)} - \sqrt{K}\}^{-1}. \quad (14)$$

Even simpler formulas result when the amplitude and phase quadratures are symmetric ($V_1^+ = V_1^- = V_1$ and $V_2^+ = V_2^- = V_2$); in this case,

$$F_c(V_1, V_2) = \frac{4V_1V_2}{(V_1 + V_2)^2}, \quad (15)$$

$$F_q(V_1, V_2) = 2\{V_1V_2 + 1 - \sqrt{(V_1^2 - 1)(V_2^2 - 1)}\}^{-1}, \quad (16)$$

where Eqs. (16) and (3) are identical.

In general, the two states can be separated by some distance $x = x_r + ix_i = \delta_2 - \delta_1$ in phase space. This separation can be shown to have the following straightforward effect (see Appendix A):

$$F_q(x) = F_q(\varphi = 0)D(x),$$

$$D(x) = \exp\left[-\frac{2x_r^2}{V_1^+ + V_2^+} - \frac{2x_i^2}{V_1^- + V_2^-}\right]. \quad (17)$$

This dependence on distance turns out to be exactly the same as one obtains for classical fidelity,

$$F_c(x) = F_c(\varphi = 0)D(x), \quad (18)$$

and we therefore consider the separation in phase space no further here.

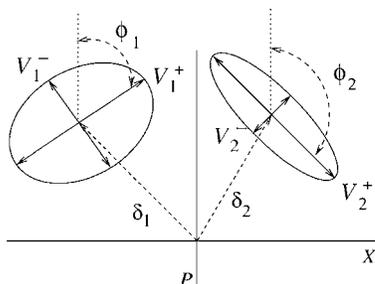


Fig. 2. Schematic of two arbitrary Gaussian distributions P_1 and P_2 .

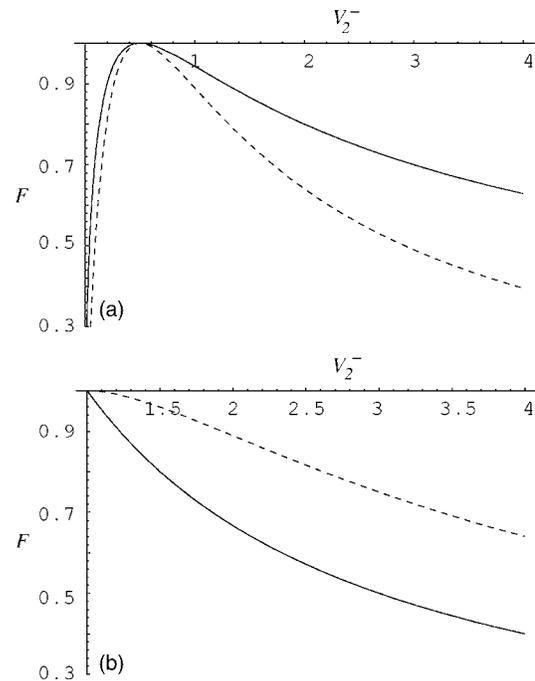


Fig. 3. Quantum (solid curves) and classical (dashed curves) fidelities F between two Gaussian distributions. (a) Both distributions are pure, $V_1^+=2$, $V_1^-=1/2$, $V_2^+V_2^-=1$, and $\varphi=0$. (b) One distribution is pure, and the other is mixed but with the same squeezing parameter $V_1^+=2$, $V_1^-=1/2$, $V_2^+/V_2^-=4$, and $\varphi=0$.

B. ρ_1 or ρ_2 Pure

Let us first compare F_c and F_q from Eqs. (11) and (12) when one of the states is pure. In this case, a simple relationship can be drawn between the quantum and classical fidelities:

$$F_q^2 = \frac{F_c}{\sqrt{V_2^+V_2^-}}. \quad (19)$$

Here, as in all following cases, when comparing classical and quantum fidelities the properties of distribution 1 are fixed while those of distribution 2 are varied. The quantum and classical fidelities between two distributions with $V_1^+=2$, $V_1^-=1/2$, $V_2^+V_2^-=1$, and $\varphi=0$ are compared in Fig. 3(a). For quantum fidelity, this condition corresponds to two pure quantum states, one of which has a varying degree of squeezing while the other has a fixed squeezing parameter of $r=-0.347$. We see that classical fidelity degrades faster than quantum fidelity. This result can be obtained straightforwardly from Eq. (19). We see that when $V_2^+V_2^-=1$, $F_c = F_q^2$, remembering the bound on fidelity $0 \leq \{F_c, F_q\} \leq 1$, it is clear that $F_c \leq F_q$.

Let us now consider the quantum and classical fidelities between a pure squeezed state and a mixed state with the same squeezing parameter. Results for the parameters $V_1^+=2$, $V_1^-=1/2$, $V_2^+/V_2^-=4$, $r=-0.347$, and $\varphi=0$ are shown in Fig. 3(b). We see that the quantum fidelity degrades faster than the classical fidelity as the difference in the mixedness of the two states increases. This trend directly contrasts with that obtained in Fig. 3(a) as the discrepancy in the squeezing parameters of the two states increases.

Figure 3(b) seems to reflect the difference between quantum and classical distributions with respect to mixedness. Since a classical ensemble consists simply of a mixture of classical states each weighted by the distribution function, it is essentially entirely mixed regardless of the breadth of the distribution. A quantum state, however, is perfectly pure if the breadth is unity ($V^+V^- = 1$) and the mixedness increases as $V^+V^- \rightarrow \infty$. So, increasing the breadth of the distribution has a secondary effect for quantum states, which is not present for classical distributions. It is reasonable, then, that differences in breadth cause a greater reduction in the similarity (and hence fidelity) of quantum states than that of their classical counterparts under the same conditions.

C. ρ_1 and ρ_2 Mixed

As we have explained in Section 2, it may not be acceptable to simply assume that the input state is pure in real teleportation experiments, since the quantum fidelity can be extremely sensitive to even small amounts of mixedness. For example, the quantum fidelity between a coherent state of $V_1=1$ and a thermal state of $V_2=2$ is $F \approx 0.667$, whereas for thermal states with $V_1=1.05$ and $V_2=2$ ($V_1=1.01$ and $V_2=2$) it is $F \approx 0.785$ ($F \approx 0.721$). If a pure input state was assumed for the latter cases, the fidelity would be only $F \approx 0.667$, a significant underestimation.

In Fig. 4 the breadths of the distributions are four times larger than those for the previous case (Fig. 3), while all other conditions are the same. The same trends as those in Fig. 3 are observed, but the differences between quantum and classical fidelities are smaller. In other words, the discrepancy between quantum and clas-

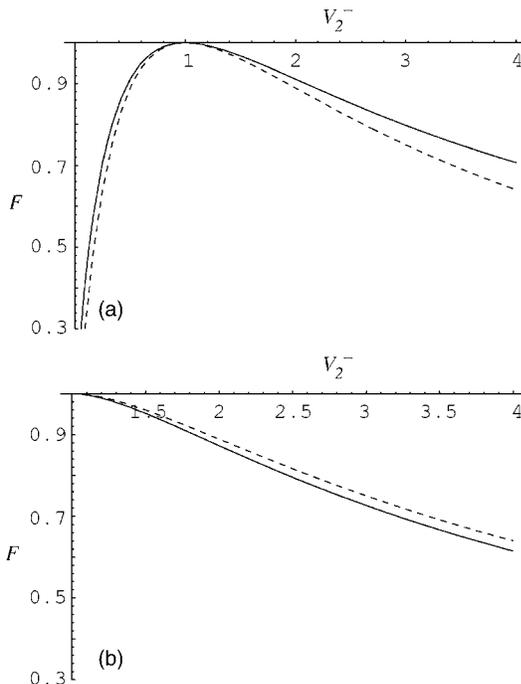


Fig. 4. Quantum (solid curves) and classical (dashed curves) fidelities between Gaussian distributions. (a) $V_1^+=4$, $V_1^-=1$, $V_2^+V_2^-=4$, and $\varphi=0$. (b) $V_1^+=4$, $V_1^-=1$, $V_2^+/V_2^-=4$, and $\varphi=0$. The breadths of the Gaussian distributions are four times larger than those in Fig. 3.

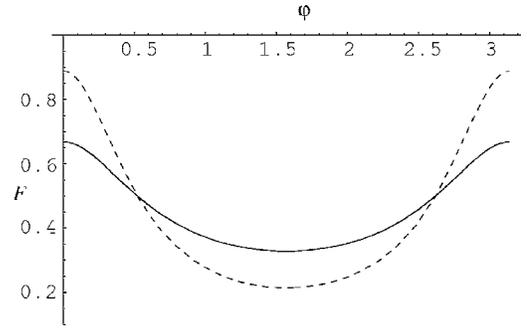


Fig. 5. Quantum (solid curve) and classical (dashed curve) fidelities for two Gaussian distributions having the same absolute squeezing parameter $V_1^+/V_1^- = V_2^+/V_2^- = 16$ ($|r|=0.693$), yet different breadths $V_1^+V_1^- = V_2^+V_2^-/4 = 1$. The quantum and classical fidelities vary differently with the relative angle.

sical fidelity is reduced as the breadth (V^+V^-) of the Gaussian distributions increases. The two fidelities become identical as $V_1^+V_1^- \rightarrow \infty$ and $V_2^+V_2^- \rightarrow \infty$. Since in this limit the quantum states can be treated as classical objects, we see that the classical limit of quantum fidelity is classical fidelity as expected.

As a final example, let us consider the effect of rotating one distribution in phase space. Suppose the two Gaussian distributions have the same absolute squeezing parameter ($V_1^+/V_1^- = V_2^+/V_2^- = 16$, $|r|=0.693$), but different breadths ($V_1^+V_1^- = V_2^+V_2^-/4 = 1$), and are initially aligned ($\varphi = 0$). In this case, as was seen previously, the difference in breadth causes the classical fidelity to be greater than the quantum fidelity [similar to Fig. 3(b)]. However, if one begins to change the relative angle φ between the distributions, the squeezing parameters r_1 and r_2 of the distributions become different. At a certain point, this difference may become more dominant than the difference in breadth [similar to Fig. 3(a)]. Thus the difference between quantum and classical fidelities gets smaller and eventually classical fidelity becomes greater as shown in Fig. 5.

In this section, we have considered several different manipulations of Gaussian states and shown that quantum and classical fidelities respond in qualitatively different manners to these manipulations. It is clear, therefore, that classical fidelity cannot be used, in general, to establish a strong physical significance for quantum fidelity. As mentioned previously, there is a clear physical significance if one of the states involved is pure. In Section 4 we propose a characterization technique that takes advantage of this fact to establish a physical significance for the fidelity of general quantum information protocols.

4. FIDELITY FOR QUANTUM INFORMATION PROTOCOLS

One of the most common applications of quantum fidelity is to measure the efficacy of quantum information protocols.¹ Typically, to characterize such protocols one begins with an ensemble of identical known input states. The protocol is then performed on each input state, yielding an ensemble of (hopefully identical) output states. These states can be fully characterized by performing tomographic measurements. The desired output state (the output state that would be achieved if the protocol ran

perfectly) from the protocol is typically well known. For example, in unity gain quantum teleportation it would simply be the input state. The fidelity between this desired output state and the actual output state can then be directly calculated and is used to judge the efficacy of the protocol. However, we have seen that fidelity is highly sensitive to both squeezing of and impurity in the states being compared. Furthermore, when both states are impure, it is difficult to attribute anything more than a weak physical significance to fidelity. Since in any realistic experiment all states involved (except perhaps vacuum states) will be at least somewhat impure, the usefulness of fidelity *per se* as a measure of the efficacy of quantum information protocols is questionable. Furthermore, since the fidelity of a quantum information protocol depends strongly on the properties of the input state, care must be taken when using fidelity to make comparisons between even very similar experiments.

At this point one would be forgiven for contemplating rejecting fidelity entirely as an efficacy measure. However, it does have one attractive feature: When at least one of the states involved is pure, it is simply the transition probability between the two states. In other words, if one were to make a projective measurement on the output state, fidelity is the probability that it would collapse into the desired output state. An alternative technique, that we advocate here, is to characterize the transfer function of the quantum information protocol. In general, this will involve characterizing the function that maps the Wigner function of the input state to that of the output state. Given particular experimental conditions and available input states, this can be achieved to the same particular precision. The fidelity for any arbitrary input state can then be calculated simply by applying the transfer function and then comparing the resulting predicted output state to the desired output state. The fidelity for some reference input state, which can be chosen arbitrarily, can then be calculated. Fidelity calculated in this way has the following advantages: (1) Since a pure input state can be chosen, the desired output state from the protocol will also often be pure;²⁵ the fidelity obtained then has physical significance as the transition probability between the desired and predicted output states. (2) Since the same reference input state can be used for all experimental implementations of quantum information protocols, this fidelity can be used as a benchmark to compare efficacies.

Let us consider, for example, how this process would work for unity gain continuous-variable quantum teleportation. As mentioned above, the desired output for unity gain continuous-variable quantum teleportation is the input state. However, as a result of imperfect entanglement and technical noise sources, the actual output state will be somewhat degraded. This effect is quantified by the transfer function of the system (if defined by the gain of the process, which due to experimental imperfections, it will not be exactly unity^{9,10}) and the additional noise present on the output state over and above the noise present on the input state. In general the additional noise is non-Gaussian, and tomographic techniques are required for a full characterization; however, when Gaussian entanglement is used and all other noise sources are Gaussian, only the variance of the noise is required.^{9,10,26}

This is the case for all continuous-variable teleportation experiments to date.^{8–11} Once the gain and noise variance have been determined, an arbitrary reference input state can be chosen and the corresponding output state calculated. A sensible choice of reference input state in this case would be a coherent state, since the classical fidelity limit for teleportation is normally quoted for coherent states. The fidelity between this reference coherent state and the output state can be directly calculated and used to compare different teleportation experiments. As we saw in Fig. 1, if this transfer function and reference state technique is not used to characterize teleportation experiments, the fidelities quoted by different experiments will vary significantly based on variations in the mixedness of the input states. This variation has no bearing on the strength of entanglement used in the experiment, or the efficacy of the protocol. Note that in most teleportation experiments to date it has simply been assumed that their input states were pure without justification. For small levels of mixedness, this is approximately (but not exactly) equivalent to what we suggest here.

For a specific example, we compare two unity gain teleportation experiments, each performed in exactly the same manner, and each with identical entanglement resources. Let us say, for argument's sake, that the entanglement is generated by interfering two squeezed beams (labeled here with subscripts *a* and *b*, respectively) with variances $V_a^+ = V_b^- = 1/V_a^- = 1/V_b^+ = 0.5$. Assuming that, apart from nonideal entanglement, both experiments are performed perfectly;²⁷ the output of each is a Gaussian state with amplitude and phase quadrature variances given by

$$V_{\text{out}}^{\pm} = V_{\text{in}}^{\pm} + 1, \quad (20)$$

where V_{in}^{\pm} are the amplitude and phase quadrature variances of the input state.^{9,10} Notice that the noise introduced to the output state is entirely independent of the input state. It should therefore be concluded that both teleportation experiments performed equally well. However, let us say that the first experiment used a coherent input state ($V_{\text{in}}^{\pm(1)} = 1$), while the second used a thermal state with $V_{\text{in}}^{\pm(2)} = 2$. The output states then have respective variances of $V_{\text{out}}^{\pm(1)} = 2$ and $V_{\text{out}}^{\pm(2)} = 3$. Substituting these values directly into Eq. (3), [$V_{\text{in}}^{\pm(i)} \rightarrow V_1, V_{\text{out}}^{\pm(i)} \rightarrow V_2$], we see that the experiments yield dramatically different fidelities of $F^{(1)} = 0.67$ and $F^{(2)} = 0.95$, and an incorrect conclusion could be drawn that experiment (2) performed much better than experiment (1).

If the fidelity is calculated via a transfer function approach, however, this difference is eliminated. As discussed previously, the transfer function of a teleportation experiment for which the entanglement and noise sources are Gaussian can be characterized simply by the teleportation gain and the variance of the introduced noise. Experimenters (1) and (2), therefore, both determine these parameters from measurements of the coherent amplitudes and variances of their respective input and output states. In both cases the gain and noise variances will be equal to unity as can be seen from Eq. (20); and Eq. (20) then directly defines the transfer function of the teleportation system. To compare experiments, the experiment-

ers choose a common reference input state, in this case a coherent state, and determine from Eq. (20) that, if such an input state was used in their system, the output variances would be $V_{\text{out}}^{\pm}=2$. They then arrive at the fidelity of teleportation for this particular reference input state from Eq. (3), which yields a value of $F=0.67$ in both cases. The experimenters therefore reach the correct conclusion that their experiments were performed equally well.

5. REMARKS

In this paper we have investigated the quantum fidelity between Gaussian states. Investigations of this kind are important, since all continuous-variable quantum information experiments to date have been performed with such states. The input states in these experiments are normally treated as pure coherent states.^{8–11} However, small levels of mixedness are typically present. We find that even these levels of mixedness significantly alter the quantum fidelity. Hence, it is typically not appropriate to simply assume that the input states are pure. In an attempt to understand why quantum fidelity is so sensitive to mixedness, and to gather some understanding of its physical significance between two mixed states, we consider its classical counterpart, the classical fidelity between two probability distributions. Since the Wigner functions of Gaussian states are positive definite, one might expect the quantum and classical fidelities to be identical or at least similar. We find, however, that they show radically different behaviors. Classical fidelity between probability distributions is degraded more strongly than quantum fidelity between quantum states as a result of differences in squeezing parameters. On the other hand, the quantum fidelity degrades faster than the classical fidelity as the breadth ($\Delta X^2 \Delta P^2$) of the distributions diverge. The distance between two Gaussian states in phase space does not cause any discrepancy between the quantum and classical fidelities. In the limit of the extreme mixedness for both Gaussian states, which can be considered the classical limit, quantum fidelity approaches the classical one.

Although a clear physical significance can be attached to quantum fidelity when one of the states involved is pure, our results indicate that when both states are mixed, quantum fidelity loses this significance. For this reason, we propose the use of transfer functions to characterize continuous-variable quantum information protocols. Once the transfer function of the protocol is determined, the fidelity that would be achieved between an arbitrary pure input state and the output state can be calculated. The resulting value has physical significance and can be used as a benchmark to compare between experiments.

APPENDIX A: QUANTUM FIDELITY FOR GAUSSIAN STATES

The density matrix of a general Gaussian state can be expressed as

$$\rho = Z(\beta)D(x)S(r)\exp\left[-\frac{\beta}{2}(aa^\dagger + a^\dagger a)\right]S^\dagger(r)D^\dagger(x), \quad (\text{A1})$$

where $S(r)$ is the squeezing operator, $D(x)$ is the displacement operator, and $Z(\beta)$ is the normalization factor. Quantum fidelity between two Gaussian states ρ_1 and ρ_2 , for $x_1=x_2$, is then²¹

$$F_q^{(\varphi)} = \frac{2 \sinh \frac{\beta_1}{2} \sinh \frac{\beta_2}{2}}{\sqrt{Y} - 1}, \quad (\text{A2})$$

where

$$Y = \cos^2 \varphi \left[\cosh^2(r_2 - r_1) \cosh^2 \frac{(\beta_1 + \beta_2)}{2} - \sinh^2(r_2 - r_1) \cosh^2 \frac{(\beta_2 - \beta_1)}{2} \right] + \sin^2 \varphi \left[\cosh^2(r_1 + r_2) \cosh^2 \frac{(\beta_1 + \beta_2)}{2} - \sinh^2(r_1 + r_2) \cosh^2 \frac{(\beta_2 - \beta_1)}{2} \right]. \quad (\text{A3})$$

The variances V^\pm for a Gaussian state of the general form of Eq. (A1) are

$$V^+ = \Delta X^2 = 1 + A + B, \quad (\text{A4})$$

$$V^- = \Delta P^2 = 1 + A - B, \quad (\text{A5})$$

$$A = 2[\bar{n} + (2\bar{n} + 1)\sinh^2 r], \quad (\text{A6})$$

$$B = 2(2\bar{n} + 1)\cosh \phi \sinh r \cosh r, \quad (\text{A7})$$

$$\bar{n} = \text{Tr}[\rho \hat{a}^\dagger \hat{a}] = \frac{1}{e^\beta - 1}, \quad (\text{A8})$$

where \bar{n} corresponds to the average photon number. Then the squeezing parameter r and inverse temperature β can be expressed in terms of $V_{1,2}^\pm$ and φ as

$$\beta = \ln \left[1 + \frac{2}{\sqrt{V^+ V^-} - 1} \right], \quad (\text{A9})$$

and Eq. (8). Equation (12) is obtained from Eqs. (8), (A9), (A2), and (A3).

Quantum fidelity between two distant Gaussian states ρ_1 and ρ_2 , for $\varphi=0$, was calculated by Wang *et al.*²² as

$$F_q^{(x)} = F_q^{(\varphi=0)} \mathcal{D}, \quad (\text{A10})$$

where

$$\begin{aligned}
\mathcal{D} &= \exp\left[\frac{(\epsilon_1 + \epsilon_2)}{\Delta}\right], \\
\Delta &= \cos \beta_1 \cosh \beta_2 \\
&\quad + \sinh \beta_1 \sinh \beta_2 \cosh 2(r_1 - r_2) - 1, \\
\epsilon_1 &= \sinh \beta_1 \sinh^2 \frac{\beta_2}{2} [(g^2 + g^{*2}) \sinh 2r_1 \\
&\quad - 2|g|^2 \cosh 2r_1], \\
\epsilon_2 &= \sinh \beta_2 \sinh^2 \frac{\beta_1}{2} [(g^2 + g^{*2}) \sinh 2r_2 \\
&\quad - 2|g|^2 \cosh 2r_2]. \tag{A11}
\end{aligned}$$

By substituting β and r in Eqs. (A10) and (A11) with Eqs. (A9) and (8), Eqs. (17) are obtained.

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council. W. P. Bowen acknowledges financial support from the Center for the Physics of Information, California Institute of Technology.

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